MATH 579: Combinatorics

Exam 3 Solutions

1. How many anagrams does $AAAABBBCCCCD$ have?

There are 4 A's, 3 B's, 4 C's, 1 D: 12 letters altogether. The number of anagrams is counted by the multinomial coefficient $\binom{12}{4,3,4,1} = \frac{12!}{4!3!4!1!} = 138600$.

2. Let $n \in \mathbb{N}_0$. Prove that $2^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k}$.

We begin with Newton's Binomial Theorem, which states that $(x+y)^n = \sum_{k\geq 0}\binom{n}{k}x^k y^{n-k}$, which applies since $n \in \mathbb{N}_0$. We take $x = y = 1$, getting $2^n = \sum_{k\geq 0}\binom{n}{k}$. However, for $k > n$, $\binom{n}{k} = 0$, so in fact the sum is finite, equalling $\sum_{k=0}^{n}\binom{n}{k}$.

3. Prove the "Hexagon Identity":
For all $k \in \mathbb{N}$, $\displaystyle\binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1} = \binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k}$.
For one bonus point, explain why it's called the Hexagon identity.

METHOD 1: We will repeatedly use $x^{\underline{a+b}} = x^{\underline{a}}(x-a)^{\underline{b}}$. $\binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1} = \frac{(n-1)^{\underline{k}}}{k!}\frac{n^{\underline{k-1}}}{(k-1)!}\frac{(n+1)^{\underline{k+1}}}{(k+1)!} = \frac{1}{(k-1)!k!(k+1)!}(n-1)^{\underline{k-1}}(n-1-(k-1))^{\underline{1}}n^{\underline{k-1}}(n+1)^{\underline{k}}(n+1-k)^{\underline{1}} = \frac{(n-1)^{\underline{k-1}}}{(k-1)!}\frac{(n+1)^{\underline{k}}}{k!}\frac{n^{\underline{k-1}}(n-(k-1))(n-k)}{(k+1)!} = \frac{(n-1)^{\underline{k-1}}}{(k-1)!}\frac{(n+1)^{\underline{k}}}{k!}\frac{n^{\underline{k}}(n-k)}{(k+1)!} = \frac{(n-1)^{\underline{k-1}}}{(k-1)!}\frac{(n+1)^{\underline{k}}}{k!}\frac{n^{\underline{k+1}}}{(k+1)!} = \binom{n-1}{k-1}\binom{n+1}{k}\binom{n}{k+1}$.

METHOD 2: We first consider the special case of $n \in \mathbb{Z}$ with $n-1 \geq k$. We have $\binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1} = \frac{(n-1)!}{k!(n-k-1)!}\frac{n!}{(k-1)!(n-k+1)!}\frac{(n+1)!}{(k+1)!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k)!}\frac{n!}{(k+1)!(n-k-1)!}\frac{(n+1)!}{k!(n-k+1)!} = \binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k}$. Now, we allow n to be a variable, and k a fixed constant. Both sides of the equation are polynomials in n, of fixed degree $k + (k+1) + (k-1) = 3k$, and they agree for infinitely many values (namely, for all $n \in \mathbb{Z}$ with $n-1 \geq k$). Hence the polynomials must be equal, i.e. the identity is proved for all n.

BONUS: It has this name because the six coefficients form a hexagon, in Pascal's triangle, around $\binom{n}{k}$.

4. Let $n \in \mathbb{N}_0$. Prove that $\displaystyle\frac{1}{n+1} = \sum_{k=0}^{n}\frac{(-1)^k}{k+1}\binom{n}{k}$.

We begin with Newton's Binomial Theorem, which states that $(x+y)^n = \sum_{k\geq 0}\binom{n}{k}x^k y^{n-k}$, which applies since $n \in \mathbb{N}_0$. This is a finite sum, since $\binom{n}{k} = 0$ for $k > n$. Setting $y = 1$, we get $(x+1)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$. Integrating both sides, we get $\frac{(x+1)^{n+1}}{n+1} = C + \sum_{k=0}^{n}\binom{n}{k}\frac{x^{k+1}}{k+1}$. We find $C = \frac{1}{n+1}$ by taking $x = 0$. Next, we take $x = -1$ to get $0 = \frac{1}{n+1} + \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k+1}}{k+1}$. We multiply both sides by -1 to get $0 = \frac{-1}{n+1} + \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+1}$, which is equivalent to what we're proving.

5. Let $m, n \in \mathbb{N}_0$. Prove that $\displaystyle\binom{m+n+1}{n} = \sum_{k=0}^{n}\binom{m+k}{k}$.

We begin with the Hockey Stick identity, which states that for all $n, k \in \mathbb{N}_0$, $\binom{n+k+1}{k+1} = \sum_{j=k}^{n+k}\binom{j}{k}$. We leave n as n, and take $k = m$, both in $\mathbb{N}_0$. This gives $\binom{n+m+1}{m+1} = \sum_{j=m}^{n+m}\binom{j}{m}$. By the symmetry identity (since $n \in \mathbb{N}_0$), $\binom{n+m+1}{m+1} = \binom{n+m+1}{(n+m+1)-(m+1)} = \binom{n+m+1}{n}$. Hence we have $\binom{n+m+1}{n} = \sum_{j=m}^{n+m}\binom{j}{m}$. Lastly we make the substitution $k = j - m$. As j varies from m to $n+m$, k varies from 0 to n. Hence we have $\binom{n+m+1}{n} = \sum_{k=0}^{n}\binom{m+k}{m}$.

6. Let $n \in \mathbb{N}_0$. Prove that $\displaystyle\binom{2n+1}{n+1} = \sum_{k=0}^{n}\binom{n}{k}\binom{n+1}{k}$.

We begin with the Chu-Vandermonde identity, which states that for $k \in \mathbb{N}_0$, $\binom{x+y}{k} = \sum_{j=0}^{k}\binom{x}{j}\binom{y}{k-j}$. Next, we take $x = n$ and $y = k = n+1$, observe that $n \in \mathbb{N}_0$, and get $\binom{2n+1}{n+1} = \sum_{j=0}^{n+1}\binom{n}{j}\binom{n+1}{(n+1)-j}$. Now, by the symmetry identity (since $n+1 \geq j$ and $n+1 \in \mathbb{Z}$), $\binom{n+1}{(n+1)-j} = \binom{n+1}{j}$, so we get $\binom{2n+1}{n+1} = \sum_{j=0}^{n+1}\binom{n}{j}\binom{n+1}{j}$. Finally, we note that the last summand is $\binom{n}{n+1}\binom{n+1}{n+1} = 0$, so we may as well omit it.